For a typical resonant circuit with $Q = 200$, the transition time is 14.6 ns when the oscillation frequency is 10 GHz. Thus it is 146 times longer than the RF period. The corresponding modulation bandwidth is

$$f_m|_{\max} \simeq \frac{1}{\Delta t} \simeq 70 \text{ MHz}.$$

Within this modulation bandwidth any modulation can be considered as the same as bias tuning. When the modulating signal is close to and above this bandwidth, the modulation sensitivity is expected to decrease. It is noted that (7) is computed under the assumption $|g_d| = (g_0/V_{RF})$. For devices with different g_d versus V_{RF} characteristics, the transition time will be different.

CONCLUSIONS

The tuning and modulation properties of transferred-electron devices have been described. The device behavior was characterized by its RF admittance which was calculated from a large-signal analysis. The bias-tuning characteristics were predicted from the device RF admittance. The tuning characteristics of the device, the circuit load, and the circuit Q factor are interrelated. Experimental results were also given and compared with the analysis. The modulation bandwidth was found to be determined by the circuit Q factor, the oscillation frequency, and the device-admittance versus RF-voltage relationship. Within this modulation bandwidth, the bias-modulation properties can be derived from the bias-tuning characteristics. The modulation sensitivity is independent of the modulating frequency, but changes with bias voltage.

ACKNOWLEDGMENT

The authors wish to thank Dr. W. R. Curtice and Prof. G. I. Haddad for valuable discussions concerning this work.

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Short Papers

The Method of Series Expansion in the Frequency Domain Applied to Multidielectric Transmission Lines

A. F. DOS SANTOS AND J. P. FIGANIER

***Abstract*—In this short paper a method of expanding the phase constant and the field of a multidielectric transmission line as a power series of the frequency is developed. The method provides a theoretical justification for the widely used "static" approximations and indicates the reason why their accuracy is frequently good. This expansion may also be useful for estimating an upper limit to the frequency band in which the dispersion does not exceed a specified value. A numerical example is included.**

Manuscript received January 6, 1975; revised April 15, 1975.
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I. INTRODUCTION

Most practical strip transmission lines are structures in which two dielectrics are present, one being air and the other a solid insulator having small losses in the operating frequency range. The solid dielectric has an electric permittivity different from that of air and therefore TEM modes cannot propagate. Independent TE and TM modes are seldom possible, and in consequence, the propagating modes in all practical two-dielectric strip lines are hybrid.

For the usual mode of operation, the lowest order one, the wave equation tends to the two-dimensional Laplace equation as the frequency tends to zero and hence the axial components should be small compared with the transverse ones for sufficiently low frequencies, making this mode almost TEM. Interest thus arises in the characteristics obtained from static fields, namely a phase constant $\beta = \omega(LC)^{1/2}$ and a characteristic impedance $Z_0 = (L/C)^{1/2}$, where L and C are, respectively, the inductance and capacitance per unit length.

In the following sections a method of field expansion as a power series of frequency is applied to solving Maxwell's equations in a lossless transmission line with an inhomogeneous dielectric. This method makes it possible to calculate, by an iterative procedure, any term of the power series from the knowledge of the static fields and can be used to compute the deviation of the characteristics of the line from their "static" approximations. Further the method provides a theoretical justification for the widely used "static" approximations and indicates the reason why the accuracy of these approximations is frequently good. The radius of convergence of the power series obtained will, in general, be finite. In any case, the method is likely to be useful in establishing the dispersion characteristics in the low-frequency range.

II. FIELD EXPANSION AS A POWER SERIES OF FREQUENCY

We shall be concerned with waves propagating along the z direction (the line axis). Maxwell's equations written in terms of transverse and axial components are

$$\begin{aligned} \nabla_t \times \bar{E}_t &= -j\omega\mu\bar{H}_z \\ \nabla_t \times \bar{H}_t &= j\omega\epsilon(\bar{r})\bar{E}_z \\ \nabla_t \times \bar{E}_z - j\beta\bar{a}_z \times \bar{E}_t &= -j\omega\mu\bar{H}_t \\ \nabla_t \times \bar{H}_z - j\beta\bar{a}_z \times \bar{H}_t &= j\omega\epsilon(\bar{r})\bar{E}_t \end{aligned} \tag{1}$$

where the common factor $\exp(j\omega t - j\beta z)$ has been omitted as usual. On the surface of each conductor the boundary conditions are

$$\bar{n} \times \bar{E}_t = 0 \quad E_z = 0. \tag{2}$$

In (1) and (2) the symbol $\epsilon(\bar{r})$ emphasizes the variation of the dielectric constant with transverse coordinates.

From the analytical form of (1) it is apparent that their solution is frequency dependent; this means that it may be possible to express it as a power series of the angular frequency ω.

Equations (1) are obtained assuming a time dependence of the form $\exp(j\omega t)$; if ω is replaced by $-\omega$, β has to be replaced by $-\beta$ to maintain the same direction of propagation. Therefore, β is represented by a series of odd terms in ω.[1] If the sign of ω is reversed, all field amplitudes remain the same apart from possible changes in sign. Hence any of these amplitudes has to be represented by either a series of even terms or a series of odd terms.

For the present transmission line problem we seek a solution leading to nonzero transverse fields at zero frequency and hence $\bar{E}_t,\bar{H}_t$ will be represented by even series. $\bar{E}_z,\bar{H}_z$ will then be represented by series of odd terms according to (1).

From the analysis of (1) it can be seen that the other possible solution ($\bar{E}_t,\bar{H}_t$ represented by odd series and $\bar{E}_z,\bar{H}_z$ by even series) is identically zero due to the fact that no static axial field components can exist.

In the following, instead of expanding the field quantities in terms of ω, the nondimensional parameter $\Omega = \omega/\omega_0$ is used, ω_0 being a convenient normalization frequency. For example the series for β and $\bar{E}_t$ take the form:

$$\begin{aligned} \beta &= b_1\Omega + b_3\Omega^3 + \cdots \\ \bar{E}_t &= \bar{E}_{t_0} + \bar{E}_{t_2}\Omega^2 + \cdots. \end{aligned}$$

Substituting into Maxwell's equations the expansions for all frequency-dependent quantities and equating terms of equal power in Ω, an infinite number of sets of equations successively coupled in an iterative way is obtained. The first two sets of equations are written as follows.

Zero-Order Equations

In the dielectric media:

$$\begin{aligned} \nabla_t \times \bar{E}_{t_0} &= 0 \\ \nabla_t \times \bar{H}_{t_0} &= 0. \end{aligned} \tag{3}$$

On the conductors:

$$\bar{n} \times \bar{E}_{t_0} = 0. \tag{4}$$

First-Order Equations

In the dielectric media:

$$\begin{aligned} \nabla_t \times \bar{E}_{z_1} - jb_1\bar{a}_z \times \bar{E}_{t_0} &= -j\omega_0\mu_0\bar{H}_{t_0} \\ \nabla_t \times \bar{H}_{z_1} - jb_1\bar{a}_z \times \bar{H}_{t_0} &= j\omega_0\epsilon(\bar{r})\bar{E}_{t_0}. \end{aligned} \tag{5}$$

On the conductors:

$$E_{z_1} = 0. \tag{6}$$

Examination of the successive sets of equations shows that any term of the series can be obtained from the knowledge of static fields by successive iterations.

III. FORMULATION IN TRANSVERSE COMPONENTS

Henceforth we restrict the analysis to the usual situation in practice, for which the dielectric medium consists of a number of homogeneous regions. In this case an alternative formulation involving only the transverse field components can be obtained either by the method of the preceding section or directly from the wave equation for harmonic fields. In fact the Helmholtz equation holds in each of the homogeneous regions and an iterative set of equations involving only transverse fields can be obtained directly from that equation. It is also assumed that the line has only two conductors.

Bearing in mind that the transverse fields are represented by series of even terms, we have

$$\nabla_t^2 \bar{A}_{t_0} = 0 \tag{7a}$$

$$\nabla_t^2 \bar{A}_{t_2} = (b_1^2 - \omega_0^2\epsilon\mu)\bar{A}_{t_0} \tag{7b}$$

where $\bar{A}$ represents one of the fields $\bar{E}$ or $\bar{H}$.

On the boundaries separating the different dielectric regions the fields are subject to the usual conditions of continuity of the tangential components. These can be completely expressed in terms of $\bar{E}_t$ or $\bar{H}_t$ taking the form:

$$\begin{gathered} \bar{E}_\tau^{(1)} = \bar{E}_\tau^{(2)} \\ \epsilon^{(1)}E_n^{(1)} = \epsilon^{(2)}E_n^{(2)} \\ \left.\frac{\partial E_n}{\partial n}\right|_1 = \left.\frac{\partial E_n}{\partial n}\right|_2 \\ \left(\frac{\partial E_n}{\partial \tau} - \frac{\partial E_\tau}{\partial n}\right)_1 = \left(\frac{\partial E_n}{\partial \tau} - \frac{\partial E_\tau}{\partial n}\right)_2 \end{gathered} \tag{8a}$$

$$\begin{gathered} \bar{H}_\tau^{(1)} = \bar{H}_\tau^{(2)} \\ H_n^{(1)} = H_n^{(2)} \\ \left.\frac{\partial H_n}{\partial n}\right|_1 = \left.\frac{\partial H_n}{\partial n}\right|_2 \\ \epsilon^{(2)}\left(\frac{\partial H_n}{\partial \tau} - \frac{\partial H_\tau}{\partial n}\right)_1 = \epsilon^{(1)}\left(\frac{\partial H_n}{\partial \tau} - \frac{\partial H_\tau}{\partial n}\right)_2 \end{gathered} \tag{8b}$$

where the symbols have the meaning given in Fig. 1. The conditions (8) are imposed on each term of the series expansion of the field.

IV. INFLUENCE OF THE FREQUENCY CHARACTERISTIC OF THE GENERATOR

It should be noted that the amplitude of the generator at a particular frequency and its frequency dependence are not known *a priori*; this is equivalent to saying that all field components are determined to within a factor which is an arbitrary function of the frequency.

[1] Only propagating modes, are considered.

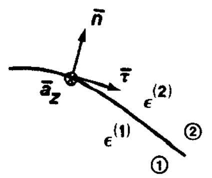

Fig. 1. Symbols used in (8).

To make the problem completely determined we may, for example, impose that the amplitude of a line integral of the field $\bar{E}_t$ between the conductors be independent of the frequency or, alternatively, we may impose the same condition on the current flowing in one of the two conductors. Obviously, we are free to impose only one of these conditions. Mathematically this is due to the fact that the two line integrals are related [see (14) and (15) in Section VI].

As far as the series expansions are concerned, the arbitrariness of the function defining the generator appears in the analysis in two ways: 1) the amplitude of either $\bar{H}_{t0}$ or $\bar{E}_{t0}$ is arbitrary; 2) an arbitrary static field satisfying the boundary conditions (8) can be added to any term of the series defining either $\bar{H}_t$ or $\bar{E}_t$. Considering, for example, the field $\bar{H}_t$, it is clear that if $\bar{H}_{t2}$ satisfies (7b) together with boundary conditions (8b) and if $\bar{H}_{t2}^{(0)}$ is a static field satisfying (8b), the sum $\bar{H}_{t2} + \bar{H}_{t2}^{(0)}$ satisfies both (7b) and (8b). But, if we impose, for example, that the current flowing in conductor 1 remain constant with frequency, $\bar{H}_{t2}^{(0)}$ is immediately determined.

The different choices of the function of the frequency defining the generator correspond to different series expansions for the field components, but the physics of the problem make it obvious that the series expansion for the phase constant must be unique.

V. QUASI-TEM LINE PARAMETERS

The zero-order equations (3) and (4) represent static fields. These fields have vanishing transverse curl and transverse divergence in the dielectric media and are nonzero due to the existence of surface charge and current on the conductors.

The first-order equations (5) together with the zero-order set, give the first approximation to the propagating field. From these equations the first terms in the power series expansion of the transmission-line parameters, are determined below.

Cross multiplying the first of equations (5) by $\bar{a}_z$, and integrating along a path ending on the conductors (Fig. 2), we obtain

$$b_1 \int_{AB} \bar{E}_{t0} \cdot \bar{dl} = \omega_0 \mu \int_{AB} H_n dl.$$

Here H_n is the component of $\bar{H}_{t0}$ perpendicular to the integration path. The first integral is equal to the voltage between the conductors. The second integral is identified as the magnetic flux per unit length. Hence

$$b_1 V_0 = \omega_0 L_0 I_0 \tag{9}$$

in which L_0 is the static inductance per unit length.

From the second of equations (5) but now using a path enclosing a conductor (Fig. 2) we obtain in a similar way the relation

$$b_1 I_0 = \omega_0 C_0 V_0 \tag{10}$$

where C_0 is the static capacitance per unit length. Multiplication of (9) by (10) yields

$$b_1 = \omega_0 (L_0 C_0)^{1/2}. \tag{11}$$

From (9) and (10) the following expression for the characteristic impedance is obtained:

$$Z_0 = \frac{V_0}{I_0} = \left(\frac{L_0}{C_0}\right)^{1/2}. \tag{12}$$

Expressions (11) and (12) provide a theoretical justification for the widely used static approximation and are in agreement with experimental results as well as with recent numerical results obtained for microstrip lines [1].

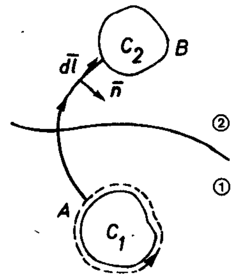

Fig. 2. Integration paths for deriving (9) and (10).

VI. HIGHER ORDER TERMS

From a practical point of view the interest in carrying out the analysis beyond the first-order terms is mainly to determine the dispersion characteristics of the transmission line. In this respect the term b_3 in the β series is the most important. In fact, an upper limit to the frequency of operation of the line can be obtained from b_3 once the maximum time delay between the components of lowest and highest frequency of the signal has been chosen.

The coefficient b_3 is obtained from the third-order equations:

$$\nabla_t H_{z3} + j(b_1 \bar{H}_{t2} + b_3 \bar{H}_{t0}) = j\omega_0 \epsilon(\bar{r}) \bar{a}_z \times \bar{E}_{t2} \tag{13a}$$

$$\nabla_t E_{z3} + j(b_1 \bar{E}_{t2} + b_3 \bar{E}_{t0}) = -j\omega_0 \mu \bar{a}_z \times \bar{H}_{t2}. \tag{13b}$$

Next we integrate (13a) along a path coinciding with the boundary of a conductor, obtaining:

$$b_1 I_2 + b_3 I_0 = \omega_0 \oint_{c1} \epsilon(\bar{r}) \bar{E}_{t2} \times \bar{a}_z \cdot \bar{dl} \tag{14}$$

where

$$I_0 = \oint_{c1} \bar{H}_{t0} \cdot \bar{dl} \quad I_2 = \oint_{c1} \bar{H}_{t2} \cdot \bar{dl}$$

are the zero-order and second-order terms in the series expansion of the current flowing in conductor C_1.

In a similar way, integration of (13b) along a path ending on the conductors yields:

$$b_1 V_2 + b_3 V_0 = \omega_0 \mu \int_{AB} \bar{H}_{t2} \times \bar{a}_z \cdot \bar{dl} \tag{15}$$

where

$$V_0 = \int_{AB} \bar{E}_{t0} \cdot \bar{dl} \quad V_2 = \int_{AB} \bar{E}_{t2} \cdot \bar{dl}.$$

It should be noted that one of the equations (14) and (15) alone is not sufficient to determine b_3, as one of these equations gives simply a relation between the line integrals of $\bar{E}_t$ and $\bar{H}_t$.

As pointed out in Section IV the series for β is unique despite the fact that the choice of the frequency dependence of a line integral of either $\bar{E}_t$ or $\bar{H}_t$ is arbitrary.

VII. NUMERICAL EXAMPLE

A numerical example is presented with a view to assess the speed of convergence of the expansion for β. A very simple structure was chosen for which the dispersion equation is known analytically thus making it possible to compare the exact values of β obtained from the dispersion equation with those corresponding to its expansion in powers of ω. This structure is a two-dielectric parallel-plane line, whose transverse section is represented in Fig. 3; it is assumed that the fields exhibit no x variations ($\partial/\partial x = 0$).

We shall consider the mode which propagates down to zero fre-

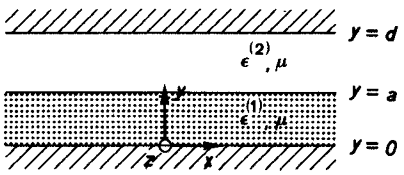

Fig. 3. Two-dielectric transmission line considered in the example of Section VII.

TABLE I

$\chi = 2$ $b_1 = 0.241840\text{E } 02$ $b_3 = 0.736682\text{ E } 02$ $b_5 = 0.964940\text{ E } 05$

f (GHz)	β (m^{-1})	R_1 (m^{-1})	R_3 (m^{-1})	R_5 (m^{-1})
0.5	12.093	0.001	0.000	0.000
1	24.191	0.007	0.000	0.000
2	48.427	0.059	0.000	0.000
3	72.753	0.201	0.002	0.000
4	97.217	0.481	0.010	0.000
5	121.870	0.950	0.029	−0.001
6	146.766	1.662	0.071	−0.004
7	171.963	2.676	0.149	−0.013
8	197.523	4.051	0.279	−0.037
9	223.504	5.848	0.478	−0.092
10	249.963	8.124	0.757	−0.208

Note: $R_1 = \beta - b_1\Omega$; $R_3 = \beta - (b_1\Omega + b_3\Omega^3)$; $R_5 = \beta - (b_1\Omega + b_3\Omega^3 + b_5\Omega^5)$.

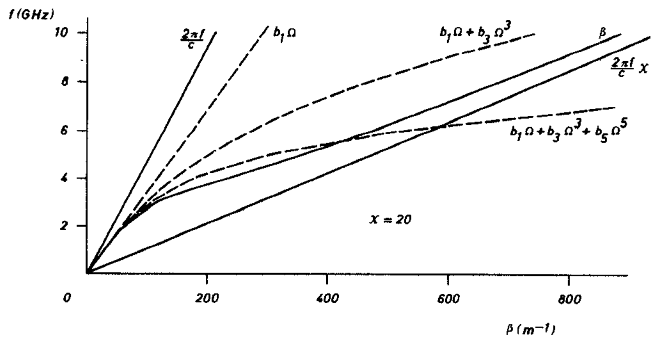

Fig. 4. Power series approximations to the phase constant of the line represented in Fig. 3.

quency; this is a TM mode with components $H_x = H$, E_y, and E_z. A standard procedure enables us to obtain the dispersion equation:

$$k^{(1)} \frac{\tan (k^{(1)}a)}{\epsilon^{(1)}} = k^{(2)} \frac{\tan [k^{(2)}(d-a)]}{\epsilon^{(2)}}$$

where superscript (1) refers to the dielectric layer $0 < y < a$ and superscript (2) to the dielectric layer $a < y < d$ and for each layer

$$k^2 = \omega^2 \epsilon \mu_0 - \beta^2.$$

Next the first three coefficients of the expansion for β, that is b_1, b_3, and b_5, were calculated, the last one with the sole purpose of assessing the speed of convergence.

The coefficients were obtained by the method developed in Sections V and VI, using the zero-order coefficient H_0 as the scale constant and imposing a frequency-independent current in the conductor $y = 0$.

The numerical computation was carried out for two cases corresponding to the following parameters:

$$d = 1 \text{ cm}$$

$$a = 0.5 \text{ cm}$$

$$\epsilon^{(2)} = \text{vacuum permittivity}$$

$$\chi = \epsilon^{(1)}/\epsilon^{(2)} = 2; 20.$$

The normalization frequency $f_0 = \omega_0/2\pi$ was taken to be 1 GHz.

For $\epsilon^{(1)}/\epsilon^{(2)} = 2$ the results are given in Table I; examination of this table shows that the error R_5 is less than 1/1000 of β within the frequency range considered.

For $\epsilon^{(1)}/\epsilon^{(2)} = 20$ the results are shown in Fig. 4; it is seen that the accuracy of the approximation degrades very quickly from the point where R_5 changes from positive to negative.

VIII. CONCLUSIONS

In the preceding sections it has been shown that for a transmission line with two conductors and a dielectric medium consisting of various homogeneous regions it is possible to expand all field functions as a power series of the frequency.

The main interest of this expansion appears to be the possibility of estimating an upper limit to the frequency band in which the dispersion does not exceed a specified value.

In this short paper the analysis has been confined to general aspects of the proposed expansion. The problem of computing the higher order terms for transmission lines of practical interest has not been considered.

ACKNOWLEDGMENT

The authors wish to thank Prof. J. Brown of Imperial College for giving the initial idea of expanding the fields as a power series of ω and for making useful comments on the manuscript.

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Useful Matrix Chain Parameter Identities for the Analysis of Multiconductor Transmission Lines

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***Abstract*—By utilizing state variable theory, certain useful matrix identities involving submatrices of the chain parameter matrix for a multiconductor transmission line are shown. These identities are extensions of familiar properties associated with two-conductor lines to multiconductor lines and are used to formulate the complete solution for the terminal currents when the line is terminated by linear networks. The identities allow a simplified solution for these currents and reduce numerous redundant time-consuming matrix multiplications. In addition, the correspondence between familiar terms for the two-conductor case and the multiconductor case is shown.**

I. INTRODUCTION

The subject of coupled transmission lines arises in the study of many microwave related structures. Transmission lines in a homogeneous medium occur in the study of strip lines whereas applica-

Manuscript received January 30, 1975; revised April 9, 1975. This work was supported by the U. S. Air Force Systems Command, Rome Air Development Center, Electromagnetic Compatibility Branch (Project 4540), Griffiss AFB, N. Y., under Contract F-30602-72-C-0418. C. R. Paul is a participant in the Postdoctoral Program, Rome Air Development Center.

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